LARGE TIME ASYMPTOTIC TO POLYNOMIALS SOLUTIONS FOR NONLINEAR DIFFERENTIAL EQUATIONS

CH. G. PHILOS, I. K. PURNARAS AND P. CH. TSAMATOS

ABSTRACT. This article is concerned with solutions that behave asymptotically like polynomials for n-th order ($n > 1$) nonlinear ordinary differential equations. For each given integer m with $1 \leq m \leq n-1$, sufficient conditions are presented in order that, for any real polynomial of degree at most m, there exists a solution which is asymptotic at ∞ to this polynomial. Conditions are also given, which are sufficient for every solution to be asymptotic at ∞ to a real polynomial of degree at most $n-1$. The application of the results obtained to the special case of second order nonlinear differential equations leads to improved versions of the ones contained in the recent paper by Lipovan [*Glasg. Math. J.* 45 (2003), 179-187] and of other related results existing in the literature.

1. INTRODUCTION

In the asymptotic theory of n-th order ($n > 1$) nonlinear differential equations, an interesting problem is that of the study of solutions with prescribed asymptotic behavior via solutions of the equation $x^{(n)} = 0$. This problem has been extensively investigated during the last four decades for the case of second order nonlinear differential equations; see Cohen [3], Constantin [4], Dannan [7], Hallam [8], Lipovan [12], Mustafa and Rogovchenko [14], Naito [15, 16, 17], Philos and Purnaras [21], Rogovchenko and Rogovchenko [25, 26], Rogovchenko [27], Rogovchenko and Villari [28], Tong [31], Yin [33], and Zhao [34] (and the references cited in these papers). For the case of linear second order differential equations, we restrict ourselves to mention the paper by Trench [32]. The above mentioned problem has also been treated for higher order nonlinear differential equations by several researchers; see Kusano and Trench [9, 10], Meng [13], Philos [18, 19, 20], Philos, Sficas and Staikos [22], Philos and Staikos [23], and the references therein. Note that the papers [18, 19, 20, 22, 23] are concerned with differential equations with deviating arguments (including the ordinary differential equations as a particular case). We also mention here the paper by Philos and Tsamatos [24] concerning nonlinear retarded differential systems. In the present paper, we are concerned with n-th order ($n > 1$) nonlinear ordinary differential equations and we study solutions that approach real polynomials of degree at most $n-1$. Our work is essentially motivated by the recent one by Lipovan [12] for the special case of second order nonlinear ordinary differential equations; the results in [12] are extended and improved in our paper.

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Consider the *n-th order* $(n > 1)$ nonlinear differential equation

$$\text{(E)} \qquad x^{(n)}(t) = f(t, x(t)), \quad t \geq t_0 > 0,$$

where *f is a continuous real-valued function on* $[t_0, \infty) \times \mathbf{R}$.

Our purpose in this paper is to investigate solutions of the differential equation (E), which behave asymptotically at ∞ like real polynomials of degree at most $n-1$, i.e. like solutions of the equation $x^{(n)} = 0$. More precisely, for each given integer m with $1 \leq m \leq n-1$, we establish sufficient conditions in order that, for any real polynomial of degree at most m, (E) has a solution defined for all large t, which is asymptotic at ∞ to this polynomial and such that the first $n-1$ derivatives of the solution are asymptotic at ∞ to the corresponding first $n-1$ derivatives of the given polynomial. We also provide conditions, which guarantee that every solution defined for all large t of (E) is asymptotic at ∞ to a real polynomial of degree at most $n-1$ (depending on the solution) and, in addition, the first $n-1$ derivatives of the solution are asymptotic at ∞ to the corresponding first $n-1$ derivatives of this polynomial. Moreover, we give sufficient conditions for every solution x defined for all large t of (E) to satisfy $x^{(n-1)}(t) \to c$ for $t \to \infty$ (and so $\left[x(t)/t^{n-1}\right] \to [c/(n-1)!]$ for $t \to \infty$), where c is some real number (depending on the solution x).

Our main results are stated in Section 2. This section contains also the application of the main results to the special case of the *second order* nonlinear differential equation

$$\text{(E}_0\text{)} \qquad x''(t) = f(t, x(t)), \quad t \geq t_0 > 0.$$

The proofs of the main results are given in Section 3. Two general examples (Examples 1 and 2) are contained in the last section (Section 4). Example 1 is concerned with the application of the main results to n-th order $(n > 1)$ Emden-Fowler equations, while Example 2 illustrates the applicability of our first main result to a specific second order superlinear Emden-Fowler equation.

We note, here, that the application of our main results to the second order nonlinear differential equation (E_0) leads to improved versions of the ones given recently by Lipovan [12] (and of other previous related results in the literature) as well as to a result due to Rogovchenko and Rogovchenko [25] (see, also, Mustafa and Rogovchenko [14]).

It is an open problem to extend the results of the present paper for the more general case of n-th order $(n > 1)$ nonlinear differential equations of the form

$$x^{(n)}(t) = F(t, x(t), x'(t), ..., x^{(n-1)}(t)), \quad t \geq t_0 > 0,$$

where F is a continuous real-valued function on $[t_0, \infty) \times \mathbf{R}^n$. This problem remains interesting still in the special case of second order nonlinear differential equations of the form

$$x''(t) = F_0(t, x(t), x'(t)), \quad t \geq t_0 > 0,$$

where F_0 is a continuous real-valued function on $[t_0, \infty) \times \mathbf{R}^2$.

2. STATEMENT OF THE RESULTS

Our main results are formulated as two theorems (Theorems 1 and 2 below), a corollary of the first of these theorems, and a proposition. Our proposition plays an important role in proving the second theorem (Theorem 2); however, it is also interesting of its own as a new result.

Throughout the paper, we are interested in solutions of the differential equation (E) which are defined for all large t, i.e. in solutions of (E) on an interval $[T, \infty)$, where $T \geq t_0$ may depend on the solution. For questions about the global existence in the future of the solutions of (E), we refer to standard classical theorems in the literature (see, for example, Corduneanu [5], Cronin [6], and Lakshmikantham and Leela [11]).

Theorem 1. *Let m be an integer with $1 \leq m \leq n-1$, and assume that*

$$|f(t,z)| \leq p(t)g\left(\frac{|z|}{t^m}\right) + q(t) \quad \text{for all } (t,z) \in [t_0, \infty) \times \mathbf{R}, \tag{2.1}$$

where p and q are nonnegative continuous real-valued functions on $[t_0, \infty)$ such that

$$\int_{t_0}^{\infty} t^{n-1}p(t)dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} t^{n-1}q(t)dt < \infty, \tag{2.2}$$

and g is a nonnegative continuous real-valued function on $[0, \infty)$ which is not identically zero.

Let $c_0, c_1, ..., c_m$ be real numbers and T be a point with $T \geq t_0$, and suppose that there exists a positive constant K so that

$$\left[\int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!}p(s)ds\right] \sup\left\{g(z) :\ 0 \leq z \leq \frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}}\right\} + \int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!}q(s)ds \leq K. \tag{2.3}$$

Then the differential equation (E) has a solution x on the interval $[T, \infty)$, which is asymptotic to the polynomial $c_0 + c_1t + ... + c_mt^m$ for $t \to \infty$, i.e.

$$x(t) = c_0 + c_1t + ... + c_mt^m + o(1) \quad \text{for } t \to \infty, \tag{2.4}$$

and, in addition, satisfies

$$x^{(j)}(t) = \sum_{i=j}^{m} i(i-1)...(i-j+1)c_it^{i-j} + o(1) \quad \text{for } t \to \infty \qquad (j = 1, ..., m) \tag{2.5}$$

and, provided that $m < n-1$,

$$x^{(k)}(t) = o(1) \quad \text{for } t \to \infty \qquad (k = m+1, ..., n-1). \tag{2.6}$$

Corollary. *Let m be an integer with $1 \leq m \leq n-1$, and assume that (2.1) is satisfied, where p and q, and g are as in Theorem 1.*

Then, for any real numbers $c_0, c_1, ..., c_m$, the differential equation (E) has a solution x on an interval $[T, \infty)$ (where $T \geq \max\{t_0, 1\}$ depends on $c_0, c_1, ..., c_m$), *which is asymptotic to the polynomial* $c_0 + c_1t + ... + c_mt^m$ *for* $t \to \infty$, *i.e.* (2.4) *holds, and, in addition, satisfies* (2.5) *and* (*provided that* $m < n-1$) (2.6).

Proposition. *Assume that*

$$|f(t,z)| \le p(t)g\left(\frac{|z|}{t^{n-1}}\right) + q(t) \quad \text{for all } (t,z) \in [t_0,\infty) \times \mathbf{R}, \tag{2.7}$$

where p *and* q *are nonnegative continuous real-valued functions on* $[t_0,\infty)$ *such that*

$$\int_{t_0}^{\infty} p(t)dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(t)dt < \infty, \tag{2.8}$$

and g *is a continuous real-valued function on* $[0,\infty)$, *which is positive and increasing on* $(0,\infty)$ *and such that*

$$\int_1^{\infty} \frac{dz}{g(z)} = \infty. \tag{2.9}$$

Then every solution x *on an interval* $[T,\infty)$, $T \ge t_0$, *of the differential equation* (E) *satisfies*

$$x^{(n-1)}(t) = c + o(1) \quad \text{for } t \to \infty \tag{2.10}$$

and

$$x(t) = \frac{c}{(n-1)!}t^{n-1} + o(t^{n-1}) \quad \text{for } t \to \infty, \tag{2.11}$$

where c *is some real number* (*depending on the solution* x).

Theorem 2. *Assume that* (2.7) *is satisfied, where* p *and* q *are as in Theorem 1, and* g *is as in Proposition.*

Then every solution x *on an interval* $[T,\infty)$, $T \ge t_0$, *of the differential equation* (E) *is asymptotic to a polynomial* $c_0 + c_1t + ... + c_{n-1}t^{n-1}$ *for* $t \to \infty$, *i.e.*

$$x(t) = c_0 + c_1t + ... + c_{n-1}t^{n-1} + o(1) \quad \text{for } t \to \infty, \tag{2.12}$$

and, in addition, satisfies

$$x^{(j)}(t) = \sum_{i=j}^{n-1} i(i-1)...(i-j+1)c_it^{i-j} + o(1) \quad \text{for } t \to \infty \qquad (j = 1,...,n-1), \tag{2.13}$$

where $c_0, c_1, ..., c_{n-1}$ *are real numbers* (*depending on the solution* x). *More precisely, every solution* x *on an interval* $[T,\infty)$, $T \ge t_0$, *of* (E) *satisfies*

$$x(t) = C_0 + C_1(t-T) + ... + C_{n-1}(t-T)^{n-1} + o(1) \quad \text{for } t \to \infty \tag{2.14}$$

and, in addition,

$$x^{(j)}(t) = \sum_{i=j}^{n-1} i(i-1)...(i-j+1)C_i(t-T)^{i-j} + o(1) \quad \text{for } t \to \infty \qquad (j = 1,...,n-1), \tag{2.15}$$

where

$$C_i = \frac{1}{i!}\left[x^{(i)}(T) + (-1)^{n-1-i}\int_T^\infty \frac{(s-T)^{n-1-i}}{(n-1-i)!}f(s,x(s))ds\right] \tag{2.16}$$
$$(i = 0, 1, ..., n-1).$$

A combination of the corollary and Theorem 2 leads to the following result:

Assume that (2.7) is satisfied, where p and q are nonnegative continuous real-valued functions on $[t_0, \infty)$ such that (2.2) holds, and g is a nonnegative continuous real-valued function on $[0, \infty)$ which is not identically zero. Then, for any real polynomial of degree at most $n-1$, the differential equation (E) has a solution defined for all large t, which is asymptotic at ∞ to this polynomial. Moreover, if, in addition, g is positive and increasing on $(0, \infty)$ and such that (2.9) holds, then every solution defined for all large t of (E) is asymptotic at ∞ to a real polynomial of degree at most $n-1$ (depending on the solution).

Now, let us concentrate on the special case of the *second order* nonlinear differential equation (E_0). In this case, Theorem 1, the corollary, the proposition, and Theorem 2 are formulated as follows:

Theorem 1A. *Assume that*

$$|f(t,z)| \leq p(t)g\left(\frac{|z|}{t}\right) + q(t) \quad \text{for all } (t,z) \in [t_0,\infty) \times \mathbf{R}, \tag{2.17}$$

where p and q are nonnegative continuous real-valued functions on $[t_0, \infty)$ such that

$$\int_{t_0}^\infty tp(t)dt < \infty \quad \text{and} \quad \int_{t_0}^\infty tq(t)dt < \infty,$$

and g is a nonnegative continuous real-valued function on $[0, \infty)$ which is not identically zero.

Let c_0, c_1 be real numbers and T be a point with $T \geq t_0$, and suppose that there exists a positive constant K so that

$$\left[\int_T^\infty (s-T)p(s)ds\right] \sup\left\{g(z) :\ 0 \leq z \leq \frac{K}{T} + \frac{|c_0|}{T} + |c_1|\right\}$$
$$+ \int_T^\infty (s-T)q(s)ds \leq K.$$

Then the differential equation (E_0) has a solution x on the interval $[T, \infty)$, which is asymptotic to the line $c_0 + c_1 t$ for $t \to \infty$, i.e.

$$x(t) = c_0 + c_1 t + o(1) \quad \text{for } t \to \infty, \tag{2.18}$$

and, in addition, satisfies

$$x'(t) = c_1 + o(1) \quad \text{for } t \to \infty. \tag{2.19}$$

Corollary A. *Assume that (2.17) is satisfied, where p and q, and g are as in Theorem 1A.*

Then, for any real numbers c_0, c_1, the differential equation (E_0) *has a solution x on an interval $[T,\infty)$ (where $T \geq \max\{t_0, 1\}$ depends on c_0, c_1), which is asymptotic to the line $c_0 + c_1 t$ for $t \to \infty$, i.e.* (2.18) *holds, and, in addition, satisfies* (2.19).

Proposition A. *Assume that* (2.17) *is satisfied, where p and q are nonnegative continuous real-valued functions on $[t_0,\infty)$ such that* (2.8) *holds, i.e. such that*

$$\int_{t_0}^{\infty} p(t)dt < \infty \quad and \quad \int_{t_0}^{\infty} q(t)dt < \infty,$$

and g is a continuous real-valued function on $[0,\infty)$, which is positive and increasing on $(0,\infty)$ and such that (2.9) *holds, i.e. such that*

$$\int_{1}^{\infty} \frac{dz}{g(z)} = \infty.$$

Then every solution x on an interval $[T,\infty)$, $T \geq t_0$, of the differential equation (E_0) *satisfies*

$$x'(t) = c + o(1) \quad for\ t \to \infty$$

and

$$x(t) = ct + o(t) \quad for\ t \to \infty,$$

where c is some real number (depending on the solution x).

Theorem 2A. *Assume that* (2.17) *is satisfied, where p and q are as in Theorem* 1A, *and g is as in Proposition* A.

Then every solution x on an interval $[T,\infty)$, $T \geq t_0$, of the differential equation (E_0) *is asymptotic to a line $c_0 + c_1 t$ for $t \to \infty$, i.e.* (2.18) *holds, and, in addition, satisfies* (2.19), *where c_0, c_1 are real numbers (depending on the solution x). More precisely, every solution x on an interval $[T,\infty)$, $T \geq t_0$, of* (E_0) *satisfies*

$$x(t) = C_0 + C_1(t-T) + o(1) \quad for\ t \to \infty$$

and, in addition,

$$x'(t) = C_1 + o(1) \quad for\ t \to \infty,$$

where

$$C_0 = x(T) - \int_{T}^{\infty} (s-T) f(s, x(s)) ds \quad and \quad C_1 = x'(T) + \int_{T}^{\infty} f(s, x(s)) ds.$$

The main results in the recent paper by Lipovan [12] are formulated as two theorems (Theorems 1 and 2). Theorem 1 in [12] is contained in Corollary A, while Theorem 2 in [12] is included in Theorem 2A. Note, also, that Proposition A has been previously established by Rogovchenko and Rogovchenko [25] (see, also, Mustafa and Rogovchenko [14]).

3. PROOFS OF THE MAIN RESULTS

In order to prove Theorem 1, we will apply the fixed point technique, by using the following Schauder's theorem (see Schauder [29]).

The Schauder theorem. *Let E be a Banach space and X any nonempty convex and closed subset of E. If S is a continuous mapping of X into itself and SX is relatively compact, then the mapping S has at least one fixed point (i.e. there exists an $x \in X$ with $x = Sx$).*

In the proof of Theorem 1, we use the Schauder theorem with $E = B([T,\infty))$, where $B([T,\infty))$ is the Banach space of all continuous and bounded real-valued functions on the interval $[T,\infty)$ endowed with the sup-norm $\|\cdot\|$:

$$\|h\| = \sup_{t \geq T} |h(t)| \quad \text{for } h \in B([T,\infty)).$$

We need the following compactness criterion for subsets of $B([T,\infty))$, which is a corollary of the Arzelà-Ascoli theorem (see Avramescu [1]; see, also, Staikos [30]).

Compactness criterion. *Let H be an equicontinuous and uniformly bounded subset of the Banach space $B([T,\infty))$. If H is equiconvergent at ∞, it is also relatively compact.*

Note that a set H of real-valued functions defined on the interval $[T,\infty)$ is called *equiconvergent at* ∞ if all functions in H are convergent in $\mathbf{R}$ at the point ∞ and, in addition, for every $\epsilon > 0$ there exists a $T' \geq T$ such that, for all functions h in H, it holds

$$\left| h(t) - \lim_{s \to \infty} h(s) \right| < \epsilon \quad \text{for all } t \geq T'.$$

To prove our proposition we will make use of the well-known Bihari's lemma (see Bihari [2]; see, also, Corduneanu [5]). This lemma is given here in a simple form which suffices for our needs.

The Bihari lemma. *Assume that*

$$h(t) \leq M + \int_{T_0}^{t} \mu(s) g(h(s)) ds \quad \text{for } t \geq T_0,$$

where M is a positive constant, h and μ are nonnegative continuous real-valued functions on $[T_0,\infty)$, and g is a continuous real-valued function on $[0,\infty)$, which is positive and increasing on $(0,\infty)$ and such that

$$\int_{1}^{\infty} \frac{dz}{g(z)} = \infty.$$

Then

$$h(t) \leq G^{-1}\left(G(M) + \int_{T_0}^{t} \mu(s) ds \right) \quad \text{for } t \geq T_0,$$

where G is a primitive of $1/g$ on $(0,\infty)$ and G^{-1} is the inverse function of G.

Now, we are in the position to proceed with the proofs of our main results.

Proof of Theorem 1. The substitution
$$y(t) = x(t) - (c_0 + c_1 t + ... + c_m t^m)$$
transforms the differential equation (E) into the equation

$$(\mathrm{E}^*) \qquad y^{(n)}(t) = f\left(t, y(t) + \sum_{i=0}^{m} c_i t^i\right), \quad t \geq t_0 > 0.$$

We immediately see that
$$y^{(j)}(t) = x^{(j)}(t) - \sum_{i=j}^{m} i(i-1)...(i-j+1)c_i t^{i-j} \quad (j = 1, ..., m)$$
and, provided that $m < n-1$,
$$y^{(k)}(t) = x^{(k)}(t) \quad (k = m+1, ..., n-1).$$
Hence, by taking into account (2.4), (2.5) and (2.6), we conclude that all we have to prove is that the differential equation (E^*) has a solution y on the interval $[T, \infty)$ with

$$(3.1) \qquad \lim_{t\to\infty} y^{(\rho)}(t) = 0 \quad (\rho = 0, 1, ..., n-1).$$

Consider the Banach space $E = B([T, \infty))$ endowed with the sup-norm $\|\cdot\|$, and define
$$Y = \{y \in E : \ \|y\| \leq K\}.$$
Clearly, Y is a nonempty convex and closed subset of E.

Now, let y be an arbitrary function in Y. For every $t \geq T$, we have
$$\frac{\left|y(t) + \sum_{i=0}^{m} c_i t^i\right|}{t^m} \leq \frac{|y(t)|}{t^m} + \sum_{i=0}^{m} \frac{|c_i|}{t^{m-i}} \leq \frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}}.$$
Consequently
$$g\left(\frac{\left|y(t) + \sum_{i=0}^{m} c_i t^i\right|}{t^m}\right) \leq \Theta \quad \text{for every } t \geq T,$$
where
$$\Theta \equiv \Theta(c_0, c_1, ..., c_m; T; K) = \sup\left\{g(z) : \ 0 \leq z \leq \frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}}\right\}.$$
On the other hand, (2.1) gives
$$\left|f\left(t, y(t) + \sum_{i=0}^{m} c_i t^i\right)\right| \leq p(t) g\left(\frac{\left|y(t) + \sum_{i=0}^{m} c_i t^i\right|}{t^m}\right) + q(t) \quad \text{for } t \geq T.$$
So, it follows that

$$(3.2) \qquad \left|f\left(t, y(t) + \sum_{i=0}^{m} c_i t^i\right)\right| \leq \Theta p(t) + q(t) \quad \text{for all } t \geq T.$$

Thus, in view of (2.2), we conclude that
$$\int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \quad \text{exists in } \mathbf{R}$$

and, more generally,

$$\int_T^\infty \frac{(s-T)^{n-1-\rho}}{(n-1-\rho)!} f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right) ds \quad \text{exists in } \mathbf{R} \qquad (\rho = 0, 1, ..., n-1).$$

Furthermore, by using (3.2), we obtain for every $t \geq T$,

$$\begin{aligned}
&\left|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right) ds\right| \\
&\leq \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right)\right| ds \\
&\leq \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} \left| f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right)\right| ds \\
&\leq \Theta \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} p(s) ds + \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} q(s) ds.
\end{aligned}$$

Hence, by taking into account (2.3), we have

$$\left|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right) ds\right| \leq K \quad \text{for every } t \geq T. \tag{3.3}$$

As (3.3) is true for any function $y \in Y$, we can immediately conclude that the formula

$$(Sy)(t) = (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right) ds \quad \text{for } t \geq T$$

defines a mapping S of Y into itself. We shall prove that this mapping satisfies the assumptions of the Schauder theorem.

First, we will show that SY is relatively compact. Since $SY \subseteq Y$, it follows immediately that SY is uniformly bounded. Moreover, for each function y in Y, we can use (3.2) to derive for all $t \geq T$

$$\begin{aligned}
|(Sy)(t) - 0| &= \left|\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right) ds\right| \\
&\leq \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right)\right| ds \\
&\leq \Theta \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} p(s) ds + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} q(s) ds.
\end{aligned}$$

So, by taking into account (2.2), we can easily verify that SY is equiconvergent at ∞. Furthermore, by using again (3.2), for any $y \in Y$ and every t_1, t_2 with

$T \le t_1 < t_2$, we get

$$\begin{aligned}
|(Sy)(t_2) - (Sy)(t_1)| &= \left| \int_{t_2}^{\infty} \frac{(s-t_2)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \right. \\
&\qquad \left. - \int_{t_1}^{\infty} \frac{(s-t_1)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \right| \\
&= \left| \int_{t_2}^{\infty} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \right] dr \right. \\
&\qquad \left. - \int_{t_1}^{\infty} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \right] dr \right| \\
&= \left| - \int_{t_1}^{t_2} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds \right] dr \right| \\
&\le \int_{t_1}^{t_2} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} \left| f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) \right| ds \right] dr \\
&\le \Theta \int_{t_1}^{t_2} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} p(s) ds \right] dr \\
&\qquad + \int_{t_1}^{t_2} \left[\int_{r}^{\infty} \frac{(s-r)^{n-2}}{(n-2)!} q(s) ds \right] dr.
\end{aligned}$$

Thus, because of (2.2), it follows that SY is equicontinuous. By the given compactness criterion, SY is relatively compact.

Next, we shall prove that the mapping S is continuous. Let $y \in Y$ and $(y_\nu)_{\nu \ge 1}$ be an arbitrary sequence in Y with

$$\lim_{\nu \to \infty} y_\nu = y.$$

By (3.2), we have

$$\left| f\left(t, y_\nu(t) + \sum_{i=0}^{m} c_i t^i\right) \right| \le \Theta p(t) + q(t) \quad \text{for every } t \ge T \text{ and for all } \nu \ge 1$$

and hence, by taking into account (2.2), we can apply the Lebesgue dominated convergence theorem to obtain, for each $t \ge T$,

$$\begin{aligned}
\lim_{\nu \to \infty} \int_{t}^{\infty} & \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y_\nu(s) + \sum_{i=0}^{m} c_i s^i\right) ds \\
&= \int_{t}^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^{m} c_i s^i\right) ds.
\end{aligned}$$

So, we have the pointwise convergence

$$\lim_{\nu \to \infty} (Sy_\nu)(t) = (Sy)(t) \quad \text{for } t \ge T.$$

It remains to verify that the convergence is also uniform, i.e.

$$\lim_{\nu \to \infty} Sy_\nu = Sy. \tag{3.4}$$

To this end, let us consider an arbitrary subsequence $(Sy_{\mu_\nu})_{\nu\geq 1}$ of $(Sy_\nu)_{\nu\geq 1}$. Since SY is relatively compact, there exists a subsequence $\left(Sy_{\mu_{\lambda_\nu}}\right)_{\nu\geq 1}$ of $(Sy_{\mu_\nu})_{\nu\geq 1}$ and a $u \in E$ so that

$$\lim_{\nu\to\infty} Sy_{\mu_{\lambda_\nu}} = u.$$

As the uniform convergence implies the pointwise convergence to the same limit function, we always have $u = Sy$. We have thus proved that (3.4) holds. Consequently, S is continuous.

Finally, the Schauder theorem implies that there exists a $y \in Y$ with $y = Sy$, i.e.

$$y(t) = (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right) ds \quad \text{for every } t \geq T.$$

Then we immediately obtain

$$y^{(n)}(t) = f\left(t, y(t) + \sum_{i=0}^m c_i t^i\right) \quad \text{for all } t \geq T,$$

which means that y is a solution on the interval $[T, \infty)$ of the differential equation (E^*). We also have

$$y^{(\rho)}(t) = (-1)^{n-\rho} \int_t^\infty \frac{(s-t)^{n-1-\rho}}{(n-1-\rho)!} f\left(s, y(s) + \sum_{i=0}^m c_i s^i\right) ds \quad \text{for all } t \geq T$$
$$(\rho = 0, 1, ..., n-1)$$

and consequently the solution y satisfies (3.1).

The proof of the theorem is complete.

Proof of the corollary. Let $c_0, c_1, ..., c_m$ be given real numbers. Consider a positive constant K so that

$$\Theta_0 \equiv \sup\left\{g(z):\ 0 \leq z \leq K + \sum_{i=0}^m |c_i|\right\} > 0.$$

(Such a K exists because of the hypothesis that g is not identically zero on $[0, \infty)$.) By (2.2), we can choose a point $T \geq \max\{t_0, 1\}$ such that

$$\int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} p(s) ds \leq \frac{K}{2\Theta_0} \quad \text{and} \quad \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} q(s) ds \leq \frac{K}{2}.$$

Since $T \geq 1$, we have

$$\frac{K}{T^m} + \sum_{i=0}^m \frac{|c_i|}{T^{m-i}} \leq K + \sum_{i=0}^m |c_i|$$

and consequently

$$\Theta \equiv \sup\left\{g(z):\ 0 \leq z \leq \frac{K}{T^m} + \sum_{i=0}^m \frac{|c_i|}{T^{m-i}}\right\}$$
$$\leq \sup\left\{g(z):\ 0 \leq z \leq K + \sum_{i=0}^m |c_i|\right\} \equiv \Theta_0.$$

Thus, we obtain

$$\left[\int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!}p(s)ds\right]\Theta + \int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!}q(s)ds \leq \frac{K}{2\Theta_0}\Theta + \frac{K}{2} \leq K,$$

i.e. (2.3) is satisfied. So, the corollary follows immediately from Theorem 1.

Proof of the proposition. Let x be a solution on an interval $[T,\infty)$, $T \geq t_0$, of the differential equation (E). Then (E) gives

$$x(t) = \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!}x^{(i)}(T) + \int_T^t \frac{(t-s)^{n-1}}{(n-1)!}f(s,x(s))ds \quad \text{for } t \geq T.$$

Thus, by using (2.7), we obtain for every $t \geq T$

$$\begin{aligned}
|x(t)| &\leq \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!}\left|x^{(i)}(T)\right| + \int_T^t \frac{(t-s)^{n-1}}{(n-1)!}\left|f(s,x(s))\right| ds \\
&\leq \sum_{i=0}^{n-1} \frac{t^i}{i!}\left|x^{(i)}(T)\right| + t^{n-1}\int_T^t \frac{1}{(n-1)!}\left[p(s)g\left(\frac{|x(s)|}{s^{n-1}}\right) + q(s)\right] ds \\
&\leq \left[\sum_{i=0}^{n-1} \frac{t^i}{i!}\left|x^{(i)}(T)\right| + \frac{t^{n-1}}{(n-1)!}\int_T^\infty q(s)ds\right] + t^{n-1}\int_T^t \frac{p(s)}{(n-1)!}g\left(\frac{|x(s)|}{s^{n-1}}\right) ds.
\end{aligned}$$

So, we have

$$\frac{|x(t)|}{t^{n-1}} \leq \left[\sum_{i=0}^{n-1} \frac{1}{i!t^{n-1-i}}\left|x^{(i)}(T)\right| + \frac{1}{(n-1)!}\int_T^\infty q(s)ds\right] + \int_T^t \frac{p(s)}{(n-1)!}g\left(\frac{|x(s)|}{s^{n-1}}\right) ds$$

for all $t \geq T$. Thus, because of the second assumption of (2.8), there exists a positive constant M so that

$$\frac{|x(t)|}{t^{n-1}} \leq M + \int_T^t \frac{p(s)}{(n-1)!}g\left(\frac{|x(s)|}{s^{n-1}}\right) ds \quad \text{for every } t \geq T. \tag{3.5}$$

Next, we define

$$G(z) = \int_M^z \frac{du}{g(u)} \quad \text{for } z \geq M.$$

(Note that $g(u) > 0$ for $u \geq M > 0$.) Clearly, G is a primitive of the function $1/g$ on $[M,\infty)$. We observe that $G(M) = 0$ and that G is strictly increasing on $[M,\infty)$. Moreover, we see that (2.9) implies $G(\infty) = \infty$. Thus, the range of G is equal to $[0,\infty)$. Let G^{-1} be the inverse function of G. The function G^{-1} is also strictly increasing on its domain $[0,\infty)$, and the range of G^{-1} equals to $[M,\infty)$. In view of the above observations, we can take into account (3.5) and use the Bihari lemma to obtain

$$\frac{|x(t)|}{t^{n-1}} \leq G^{-1}\left(G(M) + \int_T^t \frac{p(s)}{(n-1)!}ds\right) = G^{-1}\left(\frac{1}{(n-1)!}\int_T^t p(s)ds\right) \quad \text{for } t \geq T.$$

Hence, by taking into account the first assumption of (2.8), we get

$$\frac{|x(t)|}{t^{n-1}} \leq G^{-1}\left(\frac{1}{(n-1)!}\int_T^\infty p(s)ds\right) \quad \text{for every } t \geq T,$$

i.e. there exists a positive constant N so that

$$\frac{|x(t)|}{t^{n-1}} \leq N \quad \text{for all } t \geq T. \tag{3.6}$$

Now, by using (2.7) and (3.6), we derive

$$|f(t,x(t))| \leq p(t)g\left(\frac{|x(t)|}{t^{n-1}}\right) + q(t) \leq p(t) \sup_{0\leq z\leq N} g(z) + q(t) \quad \text{for } t \geq T.$$

Thus, because of (2.8), it follows immediately that

$$\int_T^\infty f(s,x(s))ds \quad \text{exists (as a real number).}$$

But, (E) gives

$$x^{(n-1)}(t) = x^{(n-1)}(T) + \int_T^t f(s,x(s))ds \quad \text{for } t \geq T.$$

Therefore,

$$\lim_{t\to\infty} x^{(n-1)}(t) = x^{(n-1)}(T) + \int_T^\infty f(s,x(s))ds \equiv c \in \mathbf{R},$$

i.e. (2.10) holds. Finally, by the L' Hospital rule, we obtain

$$\lim_{t\to\infty} \frac{x(t)}{t^{n-1}} = \frac{1}{(n-1)!} \lim_{t\to\infty} x^{(n-1)}(t) = \frac{c}{(n-1)!}$$

and consequently the solution x satisfies (2.11).

The proof of the proposition has been completed.

Proof of Theorem 2. Let x be a solution on an interval $[T,\infty)$, $T \geq t_0$, of the differential equation (E). We observe that (2.2) implies (2.8). Thus, as in the proof of the proposition, we conclude that there exists a positive constant N such that (3.6) holds. (Note that this conclusion can be obtained from the proposition itself, since it guarantees that $\lim_{t\to\infty} \left[x(t)/t^{n-1}\right] = C$ for some real number C.) By virtue of (2.7) and (3.6), we have

$$|f(t,x(t))| \leq p(t)g\left(\frac{|x(t)|}{t^{n-1}}\right) + q(t) \leq p(t) \sup_{0\leq z\leq N} g(z) + q(t) \quad \text{for } t \geq T.$$

So, by taking into account (2.2), we see that

$$L_i \equiv \int_T^\infty \frac{(s-T)^{n-1-i}}{(n-1-i)!} f(s,x(s))ds \quad (i = 0,1,...,n-1)$$

are real numbers.

Now, from (E) it follows that

$$x(t) = \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!} x^{(i)}(T) + \int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s))ds \quad \text{for } t \geq T. \tag{3.7}$$

For every $t \geq T$, we obtain

$$\begin{aligned}
&\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s))ds \\
&= -\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} d\left[\int_s^\infty f(r,x(r))dr\right] \\
&= \frac{(t-T)^{n-1}}{(n-1)!}\int_T^\infty f(r,x(r))dr - \int_T^t \frac{(t-s)^{n-2}}{(n-2)!}\left[\int_s^\infty f(r,x(r))dr\right] ds \\
&= \frac{(t-T)^{n-1}}{(n-1)!} L_{n-1} - \int_T^t \frac{(t-s)^{n-2}}{(n-2)!}\left[\int_s^\infty f(r,x(r))dr\right] ds.
\end{aligned}$$

Let us assume that $n > 2$. Then we derive, for $t \geq T$,

$$\begin{aligned}
&\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s))ds \\
&= \frac{(t-T)^{n-1}}{(n-1)!} L_{n-1} + \int_T^t \frac{(t-s)^{n-2}}{(n-2)!} d\left[\int_s^\infty (r-s) f(r,x(r))dr\right] \\
&= \frac{(t-T)^{n-1}}{(n-1)!} L_{n-1} - \frac{(t-T)^{n-2}}{(n-2)!}\int_T^\infty (r-T) f(r,x(r))dr \\
&\qquad + \int_T^t \frac{(t-s)^{n-3}}{(n-3)!}\left[\int_s^\infty (r-s) f(r,x(r))dr\right] ds \\
&= \frac{(t-T)^{n-1}}{(n-1)!} L_{n-1} - \frac{(t-T)^{n-2}}{(n-2)!} L_{n-2} \\
&\qquad + \int_T^t \frac{(t-s)^{n-3}}{(n-3)!}\left[\int_s^\infty (r-s) f(r,x(r))dr\right] ds.
\end{aligned}$$

If $n > 3$, then we can apply the same arguments to obtain, for every $t \geq T$,

$$\begin{aligned}
&\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s))ds \\
&= \frac{(t-T)^{n-1}}{(n-1)!} L_{n-1} - \frac{(t-T)^{n-2}}{(n-2)!} L_{n-2} + \frac{(t-T)^{n-3}}{(n-3)!} L_{n-3} \\
&\qquad - \int_T^t \frac{(t-s)^{n-4}}{(n-4)!}\left[\int_s^\infty \frac{(r-s)^2}{2!} f(r,x(r))dr\right] ds.
\end{aligned}$$

Following this procedure, we finally conclude that

$$\begin{aligned}
&\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s))ds \\
&= \frac{(t-T)^{n-1}}{(n-1)!}(-1)^0 L_{n-1} + \frac{(t-T)^{n-2}}{(n-2)!}(-1)^1 L_{n-2} + \dots + \frac{(t-T)^2}{2!}(-1)^{n-3} L_2 \\
&\qquad + \frac{(t-T)^1}{1!}(-1)^{n-2} L_1 + (-1)^{n-1}\int_T^t \left[\int_s^\infty \frac{(r-s)^{n-2}}{(n-2)!} f(r,x(r))dr\right] ds
\end{aligned}$$

for all $t \geq T$. Furthermore, we have for $t \geq T$

$$\begin{aligned}
&\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, x(s))ds \\
&= \sum_{i=1}^{n-1} \frac{(t-T)^i}{i!} (-1)^{n-1-i} L_i + (-1)^{n-1} \int_T^{\infty} \left[\int_s^{\infty} \frac{(r-s)^{n-2}}{(n-2)!} f(r, x(r))dr \right] ds \\
&\qquad - (-1)^{n-1} \int_t^{\infty} \left[\int_s^{\infty} \frac{(r-s)^{n-2}}{(n-2)!} f(r, x(r))dr \right] ds \\
&= \sum_{i=1}^{n-1} \frac{(t-T)^i}{i!} (-1)^{n-1-i} L_i + (-1)^{n-1} \int_T^{\infty} \frac{(r-T)^{n-1}}{(n-1)!} f(r, x(r))dr \\
&\qquad + (-1)^n \int_t^{\infty} \frac{(r-t)^{n-1}}{(n-1)!} f(r, x(r))dr \\
&= \sum_{i=1}^{n-1} \frac{(t-T)^i}{i!} (-1)^{n-1-i} L_i + (-1)^{n-1} L_0 + (-1)^n \int_t^{\infty} \frac{(r-t)^{n-1}}{(n-1)!} f(r, x(r))dr \\
&= \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!} (-1)^{n-1-i} L_i + (-1)^n \int_t^{\infty} \frac{(r-t)^{n-1}}{(n-1)!} f(r, x(r))dr.
\end{aligned}$$

Thus, (3.7) yields

$$x(t) = \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!} \left[x^{(i)}(T) + (-1)^{n-1-i} L_i \right] + (-1)^n \int_t^{\infty} \frac{(r-t)^{n-1}}{(n-1)!} f(r, x(r))dr$$

for all $t \geq T$. Hence, by taking into account the definition of L_i $(i = 0, 1, ..., n-1)$ as well as (2.16), we see that

$$(3.8) \quad x(t) = \sum_{i=0}^{n-1} C_i (t-T)^i + (-1)^n \int_t^{\infty} \frac{(r-t)^{n-1}}{(n-1)!} f(r, x(r))dr \quad \text{for all } t \geq T.$$

Since

$$\lim_{t \to \infty} \int_t^{\infty} \frac{(r-t)^{n-1}}{(n-1)!} f(r, x(r))dr = 0,$$

it follows from (3.8) that the solution x satisfies (2.14). Moreover, from (3.8) we obtain

$$\begin{aligned}
(3.9) \quad x^{(j)}(t) &= \sum_{i=j}^{n-1} i(i-1)...(i-j+1) C_i (t-T)^{i-j} \\
&+ (-1)^{n-j} \int_t^{\infty} \frac{(r-t)^{n-1-j}}{(n-1-j)!} f(r, x(r))dr \quad \text{for } t \geq T \qquad (j = 1, ..., n-1).
\end{aligned}$$

Thus, in view of the fact that

$$\lim_{t \to \infty} \int_t^{\infty} \frac{(r-t)^{n-1-j}}{(n-1-j)!} f(r, x(r))dr = 0 \qquad (j = 1, ..., n-1),$$

(3.9) guarantees that the solution x satisfies also (2.15). Finally, it is clear that there exist real numbers $c_0, c_1, ..., c_{n-1}$ so that

$$C_0 + C_1(t-T) + ... + C_{n-1}(t-T)^{n-1} \equiv c_0 + c_1 t + ... + c_{n-1} t^{n-1}$$

and so x satisfies (2.12) and, in addition, (2.13).

The proof of the theorem is now complete.

4. EXAMPLES

Example 1. Consider the n-th order ($n > 1$) Emden-Fowler equation

$$x^{(n)}(t) = a(t)\,|x(t)|^{\gamma}\,\mathrm{sgn}x(t), \quad t \geq t_0 > 0, \tag{D}$$

where a *is a continuous real-valued function on* $[t_0, \infty)$ *and* γ *is a positive real number.*

An application of Theorem 1 to the differential equation (D) leads to the following result: *Let* m *be an integer with* $1 \leq m \leq n-1$, *and assume that*

$$\int_{t_0}^{\infty} t^{n-1+m\gamma}\,|a(t)|\,dt < \infty. \tag{4.1}$$

Let $c_0, c_1, ..., c_m$ *be real numbers and* T *be a point with* $T \geq t_0$, *and suppose that there exists a positive constant* K *so that*

$$\left[\int_T^{\infty} \frac{(s-T)^{n-1}}{(n-1)!} s^{m\gamma}\,|a(s)|\,ds\right]\left(\frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}}\right)^{\gamma} \leq K.$$

Then (D) *has a solution* x *on the interval* $[T, \infty)$ *with the property:* (P(x)) x *is asymptotic to the polynomial* $c_0 + c_1 t + ... + c_m t^m$ *for* $t \to \infty$, *i.e.* (2.4) *holds, and, in addition, it satisfies* (2.5) *and (provided that* $m < n-1$*)* (2.6).

Also, by applying the corollary to the differential equation (D), we arrive at the next result: *Let* m *be an integer with* $1 \leq m \leq n-1$, *and assume that* (4.1) *is satisfied. Then, for any real numbers* $c_0, c_1, ..., c_m$, (D) *has a solution* x *on an interval* $[T, \infty)$ *(where* $T \geq \max\{t_0, 1\}$ *depends on* $c_0, c_1, ..., c_m$*) with the property* (P(x)).

Moreover, we can apply the proposition for the differential equation (D) to obtain the result: *If*

$$\int_{t_0}^{\infty} t^{(n-1)\gamma}\,|a(t)|\,dt < \infty \tag{4.2}$$

and $\gamma \leq 1$, *then every solution* x *on an interval* $[T, \infty)$, $T \geq t_0$, *of* (D) *satisfies* (2.10) *and* (2.11), *where* c *is some real number (depending on the solution* x*).*

Furthermore, by an application of Theorem 2 to the differential equation (D), we can be led to the following result: *Assume that*

$$\int_{t_0}^{\infty} t^{(n-1)(1+\gamma)}\,|a(t)|\,dt < \infty \tag{4.3}$$

and that $\gamma \leq 1$. *Then every solution* x *on an interval* $[T, \infty)$, $T \geq t_0$, *of* (D) *is asymptotic to a polynomial* $c_0 + c_1 t + ... + c_{n-1} t^{n-1}$ *for* $t \to \infty$, *i.e.* (2.12) *holds, and, in addition, satisfies* (2.13), *where* $c_0, c_1, ..., c_{n-1}$ *are real numbers (depending on the solution* x*). More precisely, every solution* x *on an interval* $[T, \infty)$, $T \geq t_0$, *of* (D) *satisfies* (2.14) *and, in addition,* (2.15), *where*

$$C_i = \frac{1}{i!}\left[x^{(i)}(T) + (-1)^{n-1-i} \int_T^{\infty} \frac{(s-T)^{n-1-i}}{(n-1-i)!} a(s)\,|x(s)|^{\gamma}\,\mathrm{sgn}x(s) ds\right]$$
$$(i = 0, 1, ..., n-1).$$

Now, let us consider the particular case of the Emden-Fowler equation (D) with

$$a(t) = t^{\lambda}\mu(t) \quad \text{for } t \geq t_0,$$

where λ is a real number and μ is a continuous and bounded real-valued function on $[t_0, \infty)$. In this case, we have

$$|a(t)| \leq \theta t^{\lambda} \quad \text{for every } t \geq t_0,$$

where θ is a positive constant. We immediately see that (4.1) is satisfied if $\lambda < -(n + m\gamma)$. Moreover, we observe that (4.2) holds if $\lambda < -[1 + (n-1)\gamma]$, while (4.3) is fulfilled if $\lambda < -[1 + (n-1)(1+\gamma)]$.

Example 2. Consider the second order superlinear Emden-Fowler equation

$$\text{(d)} \qquad x''(t) = a(t)[x(t)]^2 \operatorname{sgn} x(t), \quad t \geq t_0 > 0,$$

where a *is a continuous real-valued function on* $[t_0, \infty)$.

By applying Theorem 1 (or, in particular, Theorem 1A) to the differential equation (d), we are led to the following result: *Assume that*

$$\text{(4.4)} \qquad \int_{t_0}^{\infty} t^3 |a(t)|\, dt < \infty.$$

Let c_0, c_1 *be real numbers and* T *be a point with* $T \geq t_0$, *and suppose that there exists a positive constant* K *so that*

$$\text{(4.5)} \qquad A(T)\left(\frac{K}{T} + \frac{|c_0|}{T} + |c_1|\right)^2 \leq K,$$

where

$$\text{(4.6)} \qquad A(T) = \int_{T}^{\infty} (s-T)s^2 |a(s)|\, ds.$$

Then (d) *has a solution* x *on the interval* $[T, \infty)$, *which is asymptotic to the line* $c_0 + c_1 t$ *for* $t \to \infty$, *i.e.*

$$\text{(4.7)} \qquad x(t) = c_0 + c_1 t + o(1) \quad \textit{for } t \to \infty,$$

and, in addition, satisfies

$$\text{(4.8)} \qquad x'(t) = c_1 + o(1) \quad \textit{for } t \to \infty.$$

Now, assume that (4.4) is satisfied, and let c_0, c_1 be given real numbers and $T \geq t_0$ be a fixed point. Moreover, let $A(T)$ be defined by (4.6). In the trivial case where $A(T) = 0$, (4.5) holds by itself for any positive constant K. So, in what follows, it will be supposed that $A(T) > 0$. We easily verify that, for every positive constant K, (4.5) can equivalently be written as

$$\text{(4.9)} \qquad K^2 + 2\left[(|c_0| + |c_1|T) - \frac{T^2}{2A(T)}\right]K + (|c_0| + |c_1|T)^2 \leq 0.$$

Consider the quadratic equation

$$\Omega(\omega) \equiv \omega^2 + 2\left[(|c_0| + |c_1|T) - \frac{T^2}{2A(T)}\right]\omega + (|c_0| + |c_1|T)^2 = 0$$

in the complex plane, and let Δ be its discriminant, i.e.

$$\Delta = \left\{2\left[(|c_0| + |c_1|T) - \frac{T^2}{2A(T)}\right]\right\}^2 - 4(|c_0| + |c_1|T)^2.$$

We immediately find

$$\Delta = 4\frac{T^2}{A(T)}\left[-(|c_0| + |c_1|T) + \frac{T^2}{4A(T)}\right].$$

In the case where $\Delta < 0$, the equation $\Omega(\omega) = 0$ has no real roots and consequently $\Omega(\omega) > 0$ for all $\omega \in \mathbf{R}$. Thus, in this case, there is no positive constant K so that (4.9) is fulfilled. If $\Delta = 0$, i.e.

$$|c_0| + |c_1|T = \frac{T^2}{4A(T)}, \tag{4.10}$$

then the equation $\Omega(\omega) = 0$ has exactly one (double) real root ω_0 given by

$$\omega_0 = \frac{T^2}{4A(T)}.$$

Hence, in case that (4.10) holds, (4.9) is fulfilled (as an equality) for $K = \omega_0 > 0$, i.e. there exists a positive constant K so that (4.9) is satisfied. Next, let us consider the case where $\Delta > 0$, i.e.

$$|c_0| + |c_1|T < \frac{T^2}{4A(T)}. \tag{4.11}$$

Then the equation $\Omega(\omega) = 0$ has the real roots

$$\omega_1 = -\left(|c_0| + |c_1|T\right) + \frac{T^2}{2A(T)} - \sqrt{\frac{T^2}{A(T)}\left[-(|c_0| + |c_1|T) + \frac{T^2}{4A(T)}\right]}$$

and

$$\omega_2 = -\left(|c_0| + |c_1|T\right) + \frac{T^2}{2A(T)} + \sqrt{\frac{T^2}{A(T)}\left[-(|c_0| + |c_1|T) + \frac{T^2}{4A(T)}\right]}$$

with $\omega_1 < \omega_2$. For each real number ω, it holds

$$\Omega(\omega) \leq 0 \quad \text{if and only if} \quad \omega_1 \leq \omega \leq \omega_2.$$

By (4.11), we have

$$-\left(|c_0| + |c_1|T\right) + \frac{T^2}{2A(T)} > -\left(|c_0| + |c_1|T\right) + \frac{T^2}{4A(T)} > 0$$

and consequently ω_2 is positive. Therefore,

$$\Omega(\omega) \leq 0 \quad \text{for any } \omega \in (\max\{0, \omega_1\}, \omega_2].$$

So, provided that (4.11) is satisfied, there exists a positive constant K so that (4.9) holds. We have thus proved that there exists a positive constant K so that (4.9) (or, equivalently, (4.5)) is satisfied if and only if either (4.10) or (4.11) is fulfilled, i.e. if and only if

$$|c_0| + |c_1|T \leq \frac{T^2}{4A(T)},$$

which can equivalenlty be written as

$$A(T)\left(|c_0| + |c_1|T\right) \leq \frac{T^2}{4}. \tag{4.12}$$

We observe that (4.12) is also true if $A(T) = 0$. After the above, we can have the following result:

Assume that (4.4) *is satisfied, and let* c_0, c_1 *be real numbers and* $T \geq t_0$ *be a point so that* (4.12) *holds, where* $A(T)$ *is defined by* (4.6). *Then* (d) *has a solution* x *on the interval* $[T, \infty)$, *which satisfies* (4.7) *and* (4.8).

Finally, let us consider the particular case of the Emden-Fowler equation (d) with

$$a(t) = t^{\lambda}\mu(t) \quad \text{for } t \geq t_0,$$

where λ is a real number and μ is a continuous and bounded real-valued function on $[t_0, \infty)$. In this case, there exists a positive constant θ so that

$$|a(t)| \leq \theta t^{\lambda} \quad \text{for every } t \geq t_0.$$

We immediately see that (4.4) is satisfied if $\lambda < -4$. Furthermore, assume that $\lambda < -4$ and let c_0, c_1 be real numbers and $T \geq t_0$ be a point. Here, we have

$$A(T) \equiv \int_T^{\infty} (s - T)s^2 \, |a(s)| \, ds \leq \theta \int_T^{\infty} (s - T)s^{\lambda+2} ds = \theta \frac{T^{\lambda+4}}{(\lambda+3)(\lambda+4)}.$$

So, (4.12) is satisfied if

$$T^{\lambda+2} \left(|c_0| + |c_1| T \right) \leq \frac{(\lambda+3)(\lambda+4)}{4\theta}.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, P. O. BOX 1186, 451 10 IOANNINA, GREECE

E-mail address: cphilos@cc.uoi.gr ; ipurnara@cc.uoi.gr ; ptsamato@cc.uoi.gr